Binomial Theorem

FUN!

LEARNING :)

$$\underbrace{x+y \;,\; x+\frac{1}{x} \;,\; (x+y)^2 ,\; (x+y)^3 ,\; \left(x-\frac{1}{2x^2}\right)^n}_{\text{Binomials}}$$

Binomial Expansion

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

Pascal's Triangle

Coefficient of terms in

1

1 1

$(x+y)^2 \rightarrow$ 1 2 1

$(x+y)^3 \rightarrow$ 1 3 3 1

$(x+y)^4 \rightarrow$ 1 4 6 4 1

$(x+y)^5 \rightarrow$ 1 5 10 10 5 1

$(x+y)^6 \rightarrow$ 1 6 15 20 15 6 1

⋮

1 7 21 35 35 21 7 1

BINOMIAL THEOREM

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots$$
$${}^nC_r x^{n-r} y^r + \dots\dots\dots {}^nC_{n-1} x\, y^{n-1} + {}^nC_n y^n$$

where,
$${}^nC_0 = 1 = {}^nC_n$$
$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$
$${}^nC_r = {}^nC_{n-r}$$

Ques: Find the expansion of $(x+y)^7$

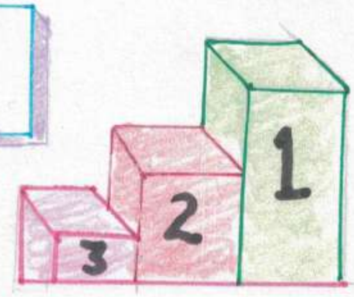
Sol:
$${}^nC_1 = {}^7C_1 = \frac{7!}{(7-1)! \times 1!} = \frac{6! \times 7}{6! \times 1} = 7$$
$${}^nC_2 = {}^7C_2 = \frac{7!}{(7-2)!\,2!} = \frac{5! \times 6 \times 7}{5! \times 2 \times 1} = 21$$
$${}^nC_3 = {}^7C_3 = \frac{7!}{(7-3)!\,3!} = \frac{4! \times 5 \times 6 \times 7}{4! \times 3 \times 2} = 35$$
$$\Rightarrow {}^nC_4 = {}^nC_3 \quad ; \quad {}^nC_5 = {}^nC_2 \quad ; \quad {}^nC_6 = {}^nC_1$$

Hence,
$$(x+y)^7 = x^7 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

Ques: Find the expansion of $\left(x - \frac{1}{x}\right)^4$

Sol: $$\left(x - \frac{1}{x}\right)^4 = x^4 - 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x}\right)^2 - 4x\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4$$

$$\left(x - \frac{1}{x}\right)^4 = x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}$$



No. of terms in the expansion of $(x+y)^n$ is $(n+1)$

$$T_r = {}^nC_{r-1}\, x^{n-r+1}\, y^{r-1}$$

Ques: Find the 4th term in the expansion of $(x+y)^{20}$?

Sol: $n = 20 \quad r = 4$

$$T_r = {}^{20}C_3\, x^{17} y^3 \qquad [n-r+1 = 20-4+1 = 17]$$

$${}^{20}C_3 = \frac{20!}{(20-3)!\,3!} = \frac{17! \times 18 \times 19 \times 20}{17! \times 3 \times 2} = 1140$$

$$\boxed{T_r = T_4 = 1140\, x^{17} y^3}$$

Ques: Find the coefficient of 6th term in the expansion of $\left(x - \frac{1}{x^2}\right)^9$?

Sol: $n = 9 \quad r = 6$

$$T_6 = {}^nC_{6-1}\, x^{n-6+1}\, y^{6-1}$$

$$T_6 = {}^9C_5\, x^4 y^5$$

$${}^9C_5 = \frac{9!}{(9-5)! \times 5!} = \frac{4! \times 5 \times 6 \times 7 \times 8 \times 9}{4! \times 2 \times 3 \times 4 \times 5}$$

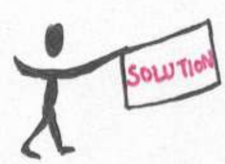
$$= 126$$

$$\boxed{\text{Coefficient of 6th term} = 126}$$

$$T_6 = 126\, x^4 \left[-\frac{1}{x^2}\right]^5$$

$$= -126\, \frac{x^4}{x^{10}}$$

$$\boxed{T_6 = -\frac{126}{x^6}}$$



Sol: 3^{rd} term from end = $(n-r+2)^{th}$ term from beginning

$= 9-3+2 = 8^{th}$

$$T_8 = {}^9C_7\, x^2 y^7$$

$$= {}^9C_7 (x^3)^2 \left[-\frac{1}{x}\right]^7$$

$$= -{}^9C_7 \frac{x^6}{x^7} = -\frac{{}^9C_7}{x}$$

$${}^9C_7 = \frac{9!}{(9-7)! \times 7!} = \frac{7! \times 8 \times 9}{2 \times 7!} = 36$$

$$\boxed{T_8 = -\frac{36}{x}}$$

Ques: Find the middle term of expansion $\left[ax - \frac{b}{x}\right]^9$

Sol: We know that expansion of $[x+y]^n$ has $(n+1)$ terms

So, $\left(ax - \frac{b}{x}\right)^9$ will have 10 terms

The expansion will have 2 middle terms:

$\left[\frac{10}{2}\right] = 5^{th}$ and $\left[\frac{10}{2}+1\right] = 6^{th}$ term

$$T_5 = {}^9C_4 (ax)^5 \left(\frac{-b}{x}\right)^4 = \frac{9!}{5! \times 4!} a^5 (b)^4 x$$

$$= \frac{6 \times 7 \times 8 \times 9}{2 \times 3 \times 4} a^5 (b)^4 x$$

$$\boxed{T_5 = 126 a^5 b^4 x}$$

$$T_6 = {}^9C_5 (ax)^4 \left(-\frac{b}{x}\right)^5 = \frac{9!}{4!5!} a^4 x^4 \cdot \frac{(-b)^5}{x^5}$$

$$\boxed{T_6 = 126 \frac{a^4 b^5}{x}}$$

Ques: If p and q be positive, then prove that coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ will be equal.

Sol: $x^p = (1)^{n-r+1}(x)^{r-1}$ $\quad$ $x^q = (1)^{n-r+1} x^{r-1}$

$p = r-1$ $\quad$ $q = r-1$

$r = p+1$ $\quad$ $r = q+1$

$T_{p+1} = {}^nC_p \ldots$ $\quad$ $T_{q+1} = {}^nC_q \ldots$

$${}^nC_p = \frac{(p+q)!}{(p+q-p)!\,p!} \qquad {}^nC_q = \frac{(p+q)!}{(p+q-q)!\,q!}$$

$$= \frac{(p+q)!}{p!\,q!} \qquad = \frac{(p+q)!}{p!\,q!}$$

Hence,

$\boxed{\text{Coefficient of } x^p = \text{coefficient of } x^q}$

Ques: Find the term independent of y in expansion of $(y^{-1/6} - y^{1/3})^9$

Sol: Term independent of y will have y^0

$$y^0 = {}^nC_r[y^{-1/6}]^{n-r+1}\ [-y^{1/3}]^{r-1}$$

$$y^0 = {}^9C_r\ [y^{-1/6}]^{10-r}\ [y^{\frac{r-1}{3}}]$$

$$y^0 = y^{\frac{-10+r}{6} + \frac{r-1}{3}}$$

$$y^0 = y^{\frac{-10+r+2r-2}{6}}$$

$0 = -12+3r$

$r = 4$

$$T_4 = -\,{}^9C_3 = -\frac{9!}{3!\,6!} = -\frac{7\times8\times9}{3\times2}$$

$\boxed{T_4 = -84}$

DETERMINANTS & MATRIX

Symbol, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ → It is called determinant of order 2 and its value is $a_1 b_2 - a_2 b_1$

$|a| = a$ → Determinant of order 1 is the no. itself.

$$\begin{array}{c} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{array}{l} \rightarrow \text{Row } R_1 \\ \rightarrow \text{Row } R_2 \\ \rightarrow \text{Row } R_3 \end{array} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \text{Column } C_1 & \text{Column } C_2 & \text{Column } C_3 \end{array} \end{array} \Rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\boxed{D = |a_{ij}| \quad \begin{array}{l} i = \text{row} \\ j : \text{column} \end{array}}$$

Value: $a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$$= a_1 (b_2 c_3 - c_2 b_3) - b_1 (a_2 c_3 - c_2 a_3) + c_1 (a_2 b_3 - a_3 b_2)$$

ARROW METHOD

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \begin{array}{ccccc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}$$

$$a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 a_2 b_1$$

Ques: $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 8 \end{vmatrix}$

Sol:
$$= 1(40-48) - 4(16-24) + 7(12-15)$$
$$= -8 + 32 - 21$$
$$= \boxed{3}$$

Ques: If $\begin{vmatrix} x & -6 & 1 \\ 2x & -3 & 4 \\ 0 & -1 & 2 \end{vmatrix} = 0 \quad ; \quad x = ?$

Sol:- $x(-6+4) + 6(4x) + 1(-2x) = 0$

$\Rightarrow \quad 22x - 2x = 0$

$$\boxed{x = 0}$$

Ques: $\begin{vmatrix} K+3 & 1 & -2 \\ 3 & -2 & 1 \\ -K & -3 & 3 \end{vmatrix} = 0 \quad ; \quad K = ?$

Solu:

$(K+3)(-6+3) - 1(9+K) - 2(-9-2K)$

$= -3K - 9 - 9 - K + 18 + 4K = 0$

$$\boxed{K \in R}$$

NOTE

1. $a_1x + b_1y + c_1 = 0$
   $a_2x + b_2y + c_2 = 0$
   $a_3x + b_3y + c_3 = 0$
   $\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

2. If $(x_1\, y_1), (x_2\, y_2), (x_3\, y_3)$ are three vertices of a Δ

   area of $\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

3. Equations of lines (concurrent)

   $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

MINORS OF ELEMENT OF A DETERMINANT

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minor of $a_{11} = m_{11}$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Similarly, $m_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

CO - FACTOR OF DETERMINANT

$$\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

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$$\Rightarrow \boxed{\sum_{i=1}^{3} a_{1i} C_{1i} = \sum_{i=1}^{3} a_{2i} c_{2i}}$$

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CIRCULAR PERMUTATION

1 Permutation of n objects (dictinct) in a circle:

$$(n-1)!$$

2 Permutation of n objects taken r at a time in a circle:

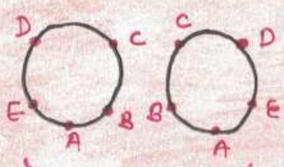
$$^nP_r$$

Ques: Find the no. of ways in which 4 people selected from 5 can be seated around a round table

Sol: $\dfrac{^5P_4}{4} = \dfrac{5!}{(5-4)!} \times \dfrac{1}{4} = \dfrac{120}{4} = \boxed{30}$

3 Arrangements of n beads in necklace

$$\frac{(n-1)!}{2}$$

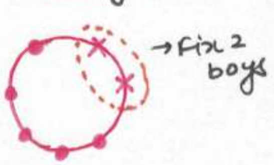


$(n-1)!$ is the no. of circular arrangements.
We will always get pairs of circular arrangement which will give one necklace.

Both are same in necklace

PRACTISE PROBLEM

1 (i) 7 boys → 2 boys should sit together



$= (6-1)!\, 2!$

$= 5!\, 2! = 120 \times 2$

$= \boxed{240}$

(ii) 7 boys

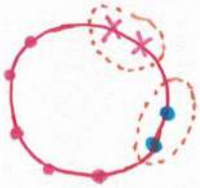
Total ways $= 6! = 120 \times 6 = 720$

Separated $= 720 - 240$

$= \boxed{480}$

2. 5 boys, 4 girls

2 girls → Not together   2 boys → not together



$(7-1)!\ 2!\ 2!$

$= 6! \times 4 = 720 \times 4 = \boxed{2880}$

Total ways $= 8! = 720 \times 7 \times 8 = \boxed{40320}$

3. 5 boys, 4 girls

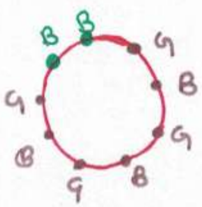
Two boys → together

$= 4!\,3!\,2!$

$= 24 \times 6 \times 2$

$= 144 \times 2$

$= \boxed{288}$



4. 13 chairs, 10 students, 1 teacher } 11 people

3 chairs → Fix

2 people - Fix

Linear way $\left\{ \begin{array}{l} 13 \text{ chairs} \\ 7 \text{ people} \end{array} \right. = {}^{13}P_7 = \dfrac{13!}{6!}$

circular $= \dfrac{13!}{6!\,7} = 8 \times 9 \times 10 \times 11 \times 12 \times 13$

No. of ways $= \dfrac{{}^nP_r}{r}$

Selections of 2 students from 10 for sitting on chairs beside Mathew $= {}^{10}C_2$

No. of ways in which 2 students can be seated beside Matthew $= {}^{10}C_2 \times 2!$

Arrangement of remaining students $= {}^{10}P_8$

$\boxed{\text{Total Arrangement} = {}^{10}C_2 \times 2! \times {}^{10}P_8}$

DIVISION INTO GROUPS/PARTITION INTO GROUPS

* Making group is different from partition

Example: A, B, C divide into (partition into) 2 groups one containing 1 object and other 2 objects.

A | BC   B | AC   C | AB → 3 ways

Example: A, B, C, D, E divide into 2 parts, one containing 2 objects and other 3 objects.

AB | CDE

BC | ADE

⋮

$$^5C_2 = \frac{5!}{2!\,3!}$$

a) Division of n object into 2 groups of size r_1 and r_2

$$^nC_{r_1} = \frac{n!}{r_1!\,(n-r_1)!} = \boxed{\frac{n!}{r_1!\,r_2!}}$$

b) Division of n object into 3 groups containing r_1, r_2, r_3 objects. $(r_1 \neq r_2 \neq r_3)$

$$= {}^nC_{r_1}\ {}^{n-r_1}C_{r_2}\ {}^{r_3}C_{r_3}$$

$$= \frac{n!}{r!\,(n-r_1)!} \times \frac{(n-r_1)!}{r_2!\,(n-r_1-r_2)!}$$

$$= \boxed{\frac{n!}{r_1!\,r_2!\,r_3!}}$$

Ques: Find the no. of ways of dividing 13 objects in 3 groups containing 8, 2 and 3 objects

Sol: $\frac{13!}{8!\,2!\,3!}$

Ques: Divide A, B, C, D into 2 group each containing 2.

Sol:

$$\underbrace{AB \mid CD \qquad AD \mid BC \qquad AC \mid BD}_{3\text{ groups}}$$

$$\frac{4!}{2!\,2!\,2!} = \frac{4\times3\times2}{2\times2\times2} = \underline{3\text{ groups}}$$

a) Making group is different from partition

ABCD

Groups

AB, BC, CD,
AD, AC, BD

$^4C_2$

Division into groups / partition

ab|cd, ac|db, ad|bc,
cd|ab, bd|ac, bc|ad

Total → 3 partitions

b) When groups are of same size, When n objects are divided into 2 groups of same size (r)

$$\boxed{\frac{n!}{r!\,r!}\times\frac{1}{2!}} \qquad (2r=n)$$

When n objects are divided into 3 groups of equal size (r)

$$\boxed{\frac{n!}{r!\,r!\,r!}\times\frac{1}{3!}}$$

c) When n object are partitioned as follows in 7 groups

→ 3 groups of size r_1 → 2 groups of size r_2

→ 1 group of size r_3 → 1 group of size r_4

$$\boxed{\frac{n!}{(r_1!)^3\times3!\;(r_2!)^2\times2!\;r_3!\,r_4!}}$$

Ques: Find the no. of ways of partition of 13 objects in 5 groups of size → 2, 2, 4, 4, 1

Sol:

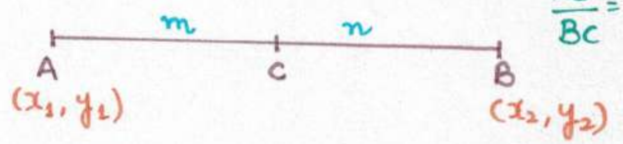
$$\frac{13!}{2!\,2!\,2!\,4!\,4!\,2!\,1!}$$

STRAIGHT LINES

RATIO CONCEPT

Internal Division

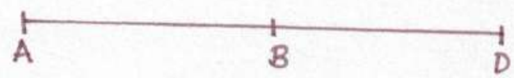
$\frac{AC}{BC} = \frac{m}{n}$



Co-ordinates of pt. C $= \frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}$

External Division

$\frac{AD}{BD} = \frac{m}{n}$



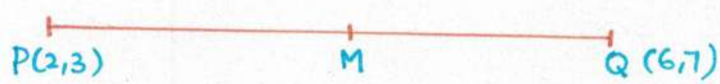
Co-ordinates of pt. D $= \frac{mx_2 - mx_1}{m-n}, \frac{my_2 - ny_1}{m-n}$

Ques: Find co-ordinates of the point M dividing P(2,3) and Q (6,7) in ratio of 1:3:

i) Internally

ii) Externally

Sol: i)

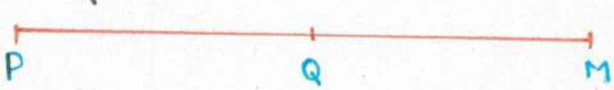


$m:n = 1:3$

Co-ordinates $= \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

$= \left(\frac{6+6}{4}, \frac{7+9}{4}\right) = (3,4)$

ii)



Co-ordinates of M $= m:n = 1:-3$

$$= \left(\frac{2(-3)+6(1)}{-3+1} , \frac{3(-3)+7}{(-3)+1} \right)$$

$$= (0,1)$$

$A(x_1, y_1)$ and $B(x_2, y_2)$ are divided in ratio by the point $C(x_0, y_0)$ is -

$$x_0 - x_1 : x_2 - x_0$$

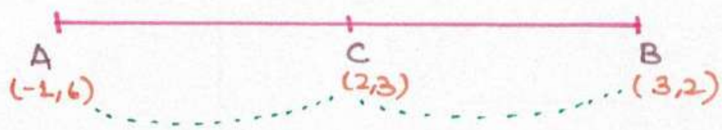
$$y_0 - y_1 : y_2 - y_0$$

If ratio is +ve ⟶ Internal division

If ratio is -ve ⟶ External division

Ques: Find ratio in which point (2,3) divides point A (1,-6) and B (3,2)

Sol:



Ratio $= 2-(-1) : 3:2$

$= 3:1$

Internal Division

OR

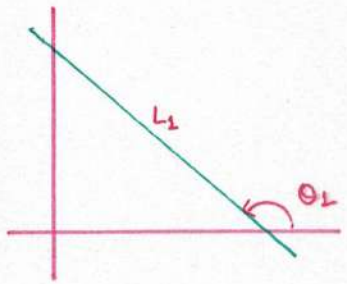
Ratio $= 3-(6) : 2-3$

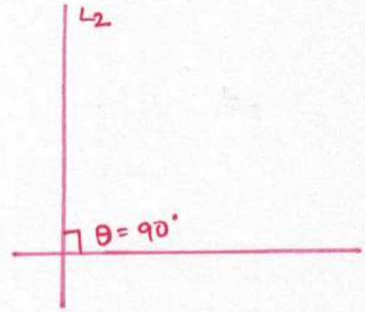
$= 3:1$

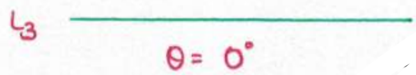
LINE

INCLINATION Angle made by the line with x- axis in anticlockwise direction (θ)

$$\boxed{\theta \in [0, \pi)}$$

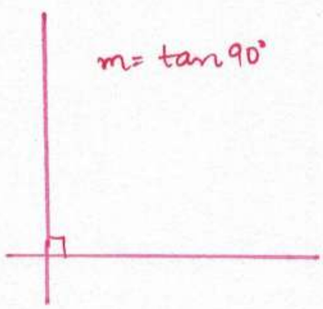


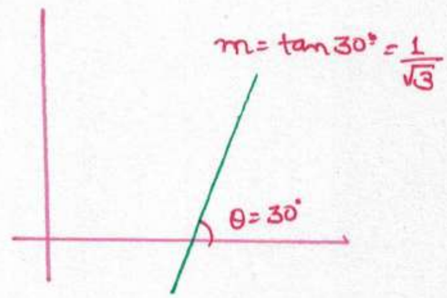




SLOPE OF A LINE

$$\boxed{m = \tan\theta}$$





Slope of a line passing through (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Ques: Find slope of the line joining $(1,2)$ and $(7,5)$

Sol: Slope $= m = \frac{3}{6} = \boxed{\frac{1}{2}}$

EQUATION OF STRAIGHT LINE:

- Requirements – (1) A point on line (2) Slope of line
- Let the line pass through the pt. (x_1, y_1)

(I) slope $= -a/b$, $ax + by = ax_1 + by_1$

(II) Slope $= a/b$, $ax - by = ax_1 - by_1$

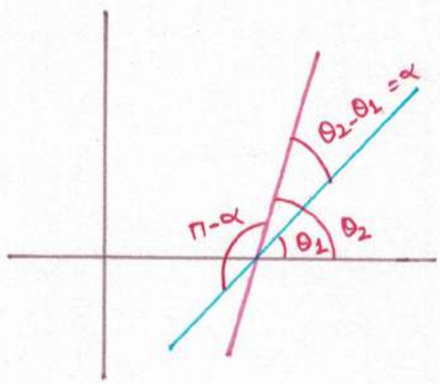
Ques: Slope = -2 , Point = $(1,2)$, Find equation?

Sol: Slope = $\frac{-2}{1} = -\frac{a}{b}$

$2x+y = 2(1) + (1)(2)$

$$\boxed{2x+y = 4}$$

ANGLE BETWEEN TWO LINES



Let, acute angle between lines $= \alpha$

$\alpha = \theta_2 - \theta_1$

$\tan\alpha = \tan(\theta_2-\theta_1)$

$$\tan\alpha = \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_2 \cdot \tan\theta_1}$$

$$\boxed{\tan\alpha = \left|\frac{m_2-m_1}{1+m_1m_2}\right|}$$ For acute $\angle$

$$\boxed{\text{Angle between two lines} = \tan^{-1}\left(\pm\frac{m_1-m_2}{1+m_1m_2}\right)}$$

FINDING SLOPE OF A LINE

eg: $2x = 3y+10$

$2x-3y = 10$

$$m = -\frac{\text{Coeff. of } x}{\text{Coeff. of } y} = -\frac{2}{-3} = \frac{2}{3}$$

OR, $y = mx+c$

$3y = 2x+10$

$y = \frac{2}{3}x + \frac{10}{3} \Rightarrow m = \frac{2}{3}$

REMEMBER

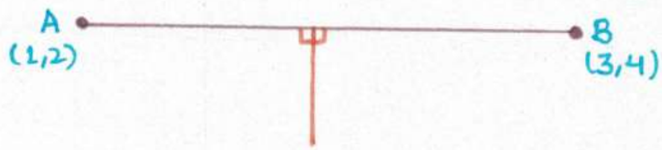
$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

A) If lines are parallel, $m_1 = m_2$

b) If lines are $\perp$ to each other, $m_1 m_2 = -1$

Ques: Find slope of the line $\perp$ to the line joining (1,2), (3,4)

Sol:



If lines are $\perp$,

$$m_1 \cdot m_2 = -1$$

slope of line AB,

$$m_1 = \frac{2}{2} = 1$$

$$\boxed{m_2 = -1}$$

PARALLEL AND PERPENDICULAR LINE STRUCTURE

$ax + by + c = 0$

	Parallel	Perpendicular
$ax + by = c$	$ax + by = \lambda$	$bx - ay = \lambda$
$ax - by = c$	$ax - by = \lambda$	$bx + ay = \lambda$

Ques: Line $\parallel$ to $5x = 7y + 6$

Ans: $5x - 7y = 6$

Line $\parallel \Rightarrow 5x - 7y = \lambda$

Ques: Line $\parallel$ to $x = -3y + 6$

Ans: $x + 3y = 6$

Line $\parallel \Rightarrow x + 3y = \lambda$

Ques: Line $\parallel$ to $x = 0$

Ans: $x = \lambda$

Ques: Line $\perp$ to $-3x+5y=7$

Ans: $3x-5y=-7$

$5x+3y=\lambda$

$m_1=\frac{3}{5}$, $m_2=\frac{-5}{3}$

$m_1 \cdot m_2=-1$ Verified

If three points given:

① If pts. are co-linear:-

(i) $AB+BC=AC$

(ii) $M_{AB}=M_{AC}$ or $M_{AB}=M_{BC}$

(iii) Ar. $(\Delta ABC)=0$

② If pts. are not co-linear ⇒ Points will frame Δ, find all sides AB, BC, CA

(i) $AB=BC=CA$ ⟶ Equilateral Δ

(ii) $AB=BC\neq CA$ ⟶ Isoscales Δ

(iii) $AB=BC\neq CA$, $CA^2=AB^2+BC^2$ ⟶ Right angle Isoscales Δ

③ $\boxed{l_1>l_2>l_3}$

(i) $l_1^2=l_2^2+l_3^2$ ⟶ Right angle Δ

(ii) $l_1^2>l_2^2+l_3^2$ ⟶ obtuse angle Δ

(iii) $l_1^2<l_2^2+l_3^2$ ⟶ Acute angle Δ

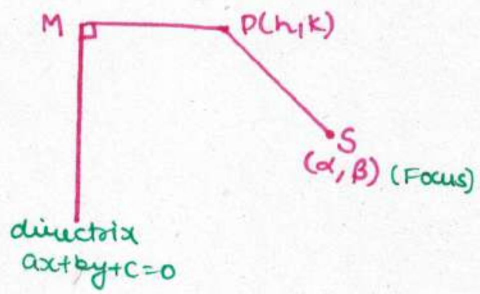
Ques: Find K if (K, K+1), (3,5), (K+2, 1) are co-linear

Sol: $\frac{5-K-1}{3-K}=\frac{1-K-1}{K+2-1}$

$K=\frac{1\pm\sqrt{33}}{2}$

parabola

DEFINATION The locus of a point which moves in such a way that the distance of the point and from a fixed point and from a fixed line is always equal.



$e = \frac{SP}{PM}$

For parabola, $e=1$

$SP = PM$

The fixed pt. is focus (s) and the fixed line is directrix. Locus is known as **Parabola**.

Eccentricity (e) is the ratio of SP and PM.

i.e. $e = \frac{SP}{PM}$

$$\boxed{(x-\alpha)^2 + (y-\beta)^2 = \left|\frac{(ax+bx+c)^2}{a^2+b^2}\right|}$$

Ques: Find the equation of a Parabola whose S(2,3) and directrix, $x+y+1=0$

Sol: $(x-2)^2 + (y-3)^2 = \frac{(x+y+1)^2}{2}$

$\Rightarrow 2(x^2+4-4x) + 2(y^2+9-6y) = x^2+y^2+1+2x+2y+2xy$

$\Rightarrow x^2+y^2-2xy-10x-14y+25=0$

> The second degree general equation → $ax^2+by^2+2hxy+2gx+2fy+c=0$ represents a parabola when,
>
> $\Delta \neq 0$ and $h^2 = ab$
>
> $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

Ques: S(a, 0), directrix → $x+a=0$

Sol: $(x-a)^2 + y^2 = (x+a)^2$

$y^2 = (x+a)^2 - (x-a)^2$

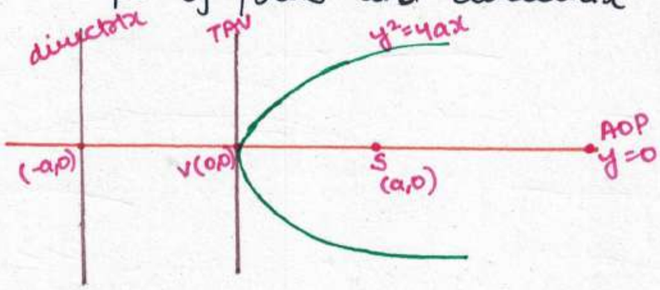
Parabola - $y^2 = 4ax$

TERMINOLOGY

- **Axis of Parabola (AOP)** - The line of symmetry which contain focus of the parabola is known as AOP.
- **Tangent at Vertex (TAV)** - A line which is $\perp r$ (perpendicular) to AOP and passing through the vertex of parabola.
- **Vertex (v)** - The POI of parabola and AOP is Vertex
- **Focus (S)** - The fixed pt. which lies on AOP is known as focus
- **Directrix (D)** - The fixed line which is $\perp r$ to AOP and is of the same distance from vertex to focus in opposite direction

$VS = VD$

vertex is the mid pt. of focus and directrix



- **Chord** - A line which cuts the parabola at 2 distinct points is known as chord
- **Focal chord** - A chord which passes through focus
- **Double Ordinate** - A line which is $\perp r$ to AOP and cuts parabola at 2 distinct points is known as double ordinate.
- **Latus Rectum** - Double ordinate passing through focus is known as LR

$\boxed{\text{Length of Latus rectum} = 4a}$

Basically, Latus rectum is also a chord, focal chord and doube ordinate to parabola.

Latus rectum - $x = a$ $\quad x+7 = 2 \quad \Rightarrow x = -5$

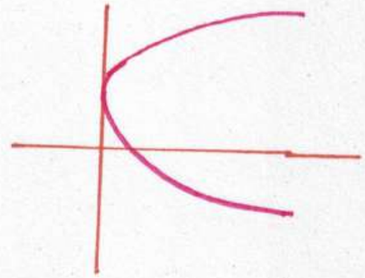
Directrix - $x = -a$ $\quad x+7 = -2 \quad \Rightarrow x = -7$

TAV - $x = 0 \quad \Rightarrow x = -7$

Ques: $y^2 - 6x - 6y + 9 = 0$

sol: $(y^2 - 2(3y) + (3)^2) = 6x$

$(y-3)^2 = 6x \quad = 4\left(\frac{6}{4}\right)x \qquad a = \frac{3}{2}$



$V = (0,0) = (0,3)$

$S = (a,0) = \left(\frac{3}{2}, 3\right)$

AOP = $y = 0 \Rightarrow y = 3$

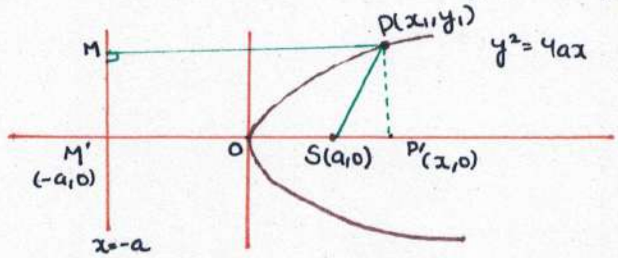
TAV = $x = 0 \Rightarrow x = 0$

Directrix $x = -a \Rightarrow x = -3/2$

LR = $x = a \Rightarrow x = 3/2$

FOCAL LENGTH

The distance of a point on Parabola from focus is known as Focal length.



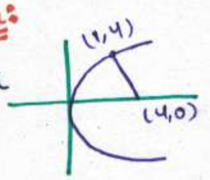
$PS = PM \Rightarrow PS = P'M' = OP' + OM'$

$PS = |x_1| + a$

> Focal Length $= |x| + a$
> $\Rightarrow$ (mode of co-ordinate x of given point) $+ a$

Example:

$y^2 = 16x$



Focal length $= |1| + 4 = 5$

or we can also apply distance formula and find the focal length.

$$d = \sqrt{(4-1)^2 + (4-0)^2} = \sqrt{16+9} = 5$$

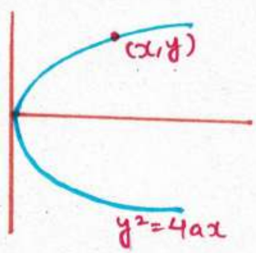
> NOTE: If the parabola is vertical, then focal length will be - $|y_1| + a$

Position of a Point w.r.t. Parabola

If we wish to find the position of a pt. w.r.t. parabola $y^2 = 4ax$ then first satisfy the pt (x_1, y_1) in the equation of parabola i.e. $y_1^2 - 4ax_1$

If $y_1^2 - 4ax_1$	> 0	Outside
$y_1^2 - 4ax_1$	$= 0$	On
$y_1^2 - 4ax_1$	< 0	Inside

Parametric Coordinates



$y^2 = 4ax$

$\frac{y}{2a} = \frac{2x}{y} = t$

t is a parameter

$y = 2at$

$2x = yt = 2at^2$

$x = at^2$

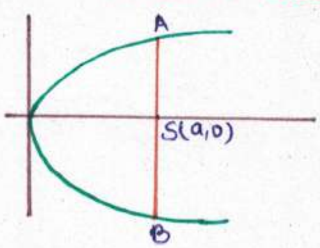
$x = at^2,\ y = 2at$ } Parametric equation of parabola

Parametric co-ordinates are-

$$\boxed{(at^2, 2at)} \qquad t \in R$$

Ques: If $A(t_1)$ and $B(t_2)$ are 2 pts. on the parabola $y^2 = 4ax$, then find the relation between t_1 and t_2 if AB is a focal chord.

Sol:



$A = (at_1^2, 2at_1) \qquad B = (at_2^2, 2at_2)$

$\Rightarrow \frac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \frac{0 - 2at_1}{a - at_1^2}$

$\Rightarrow \frac{t_2 - t_1}{t_2^2 - t_1^2} = \frac{-t_1}{1 - t_1^2}$

$\Rightarrow \frac{t_2 - t_1}{(t_2 - t_1)(t_2 + t_1)} = \frac{-t_1}{1 - t_1^2}$

$\Rightarrow t_1^2 - 1 = t_1 t_2 + t_1^2$

$\Rightarrow \boxed{t_1 t_2 = -1}$

If a parabola has a focal chord then, the coordinates of its end pts on parabola are $(at^2, 2at)$ and $(a/t^2, -2a/t)$

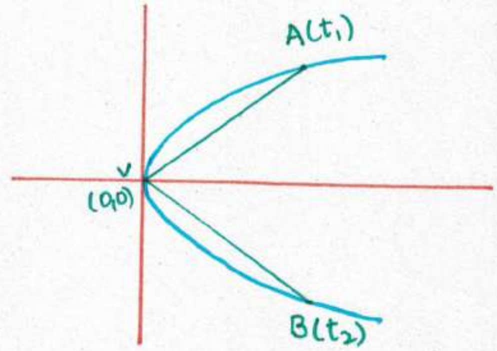
Remember → $t_1 t_2 = -1$ → For focal chord

Ques: If the line joining $A(t_1)$ and $B(t_2)$ subtend right angle at the vertex of the parabola. $y^2 = 4ax$. Then find relation between t_1 and t_2

Sol:

$$m_{AV} = \frac{2at_1}{at_1^2}$$

$$m_{BV} = \frac{2at_2}{at_2^2}$$



$\Rightarrow \quad m_{AV}\, m_{BV} = -1$

$\Rightarrow \quad \dfrac{2at_1}{at_1^2} \cdot \dfrac{2at_2}{at_2^2} = -1$

$\Rightarrow \quad \boxed{-4 = t_1 t_2}$

Note: Focal chord cannot subtend 90° at the vertex and a chord which subtend 90°, at the vertex cannot be a focal chord.

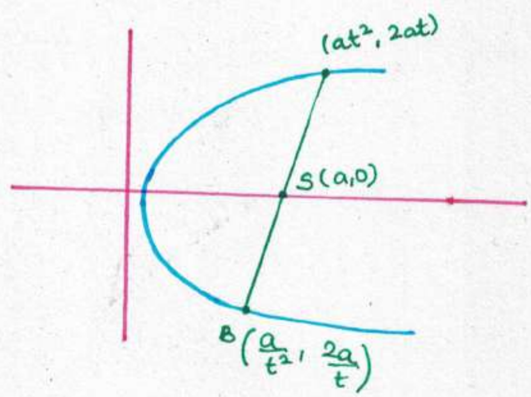
Ques: For the parabola, $y^2 = 4ax$, find the length of the focal chord in parametric form and also prove that the length of latus rectum is $4a$.

Sol: Length focal chord $= AS + BS$

$$= at^2 + a + \frac{a}{t^2} + a$$

$$= a\left(t^2 + \frac{1}{t^2} + 2\right)$$

$$= a\left(t + \frac{1}{t}\right)^2$$



For AB to be latus rectum

$at^2 = a \quad \Rightarrow t = \pm 1 \qquad t = 1$ acceptable

length of latus rectum $= a(1+1)^2 = 4a$

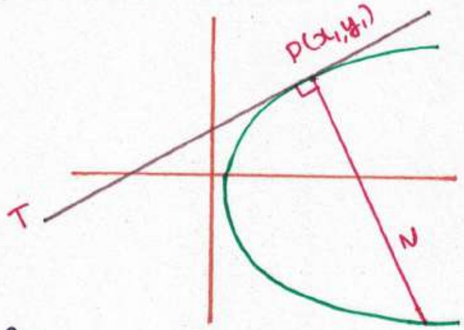
The length of focal chord to a parabola with parameter 't' is-

$$\left| a\left(t + \frac{1}{t}\right)^2 \right|$$

TANGENTS AND NORMALS

TANGENT AT THE POINT

*** POINT FORM:** Equation of tangent to the parabola $y^2 = 4ax$ at $P(x_1, y_1)$, then it will be given as $T=0$



$$YY_1 - 2a(x+x_1) = 0$$

$$\boxed{YY_1 = 2a(x+x_1)}$$

*** EQUATION OF NORMAL:**

Slope of tangent at $P(x_1, y_1) = \dfrac{2a}{y_1}$

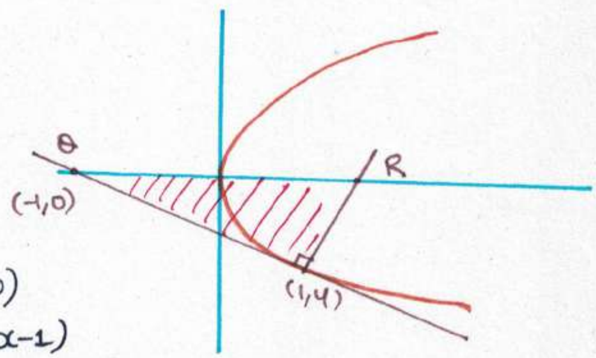
$m_1 = -y_1/2a$

Equation of normal

$$\boxed{Y - Y_1 = \left(\frac{-Y_1}{2a}\right)(x - x_1)}$$

Ques: Find the area of the Δ formed by AOP of the parabola, $y^2 = 16x$ with the tangent and normal at $(1, 4)$

Sol: Eq^n of tangent -



$y(-4) = 8(x+1)$

$-4y = 8x+8$

$8x + 4y + 8 = 0$

Eq^n of AOP -

$y = 0$, $x = -1$, $Q(-1, 0)$

Eq^n of normal $(y+4) = \dfrac{-4}{2(4)}(x-1)$

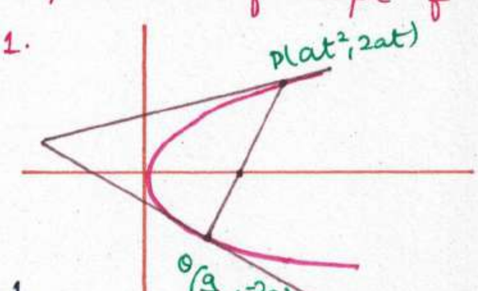
$2y + 8 = -x + 1$

$2y = -x - 7$ $\quad y = 0$

$R(9, 0)$ $\quad x = 9$ $\quad x = -7$

area $= \frac{1}{2} \times B \times 4 = \quad \frac{1}{2} \times 20 \times 2 = 20$ units

Ques: A chord is drawn inside the parabola $y^2 = 4ax$ which is passing through focus. Prove that the product of slope of the tangents drawn the extremities is -1.



Sol: Tangent at P -

$y(2at) = 2a(x + at^2)$

at Q,

$y\left(-\frac{a2}{t}\right) = 2a\left(x + \frac{a}{t^2}\right)$

$\Rightarrow m_1 \times m_2 = 2a/2at \times 2at/-2a = -1$